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Explosive instabilities in nonlinear perturbation

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Abstract. We present a general theoretical investigation of three-wave interactions by the method of nonlinear perturbation, with special emphasis on nonlinear explosive instabilities in the presence of linear damping or growth.

1. Introduction

There is a growing interest in the possibility of unbounded solutions to the equations of nonlinear interactions in plasma. The corresponding explosive instabilities may cause some astrophysical phenomena (Sturrock 1966) as well as enhanced losses in laboratory plasma (Kadomtsev *et al* 1965, Dikasov *et al* 1965, Coppi *et al* 1969). The effects of phase are of considerable importance in nonlinear wave interactions in plasma (Engelmann and Wilhelmsson 1969), and the coupled equations for three-wave nonlinear equations can be integrated analytically when the coupling constants are real. However, in the general case of an explosive instability where one takes the coupling coefficients to be complex, with phases not equal to 0, π, the problem becomes considerably more complicated (Wilhelmsson and Stenflo 1970). It has been pointed out that in the nonlinear perturbation developed by Coffey and Ford (1969) and others, Case (1966) has the distinct advantage of separating a given motion into a secular motion plus a rapidly fluctuating motion of small amplitude. In the present paper we discuss, within the framework of nonlinear perturbation theory, how a nonlinear three-wave interaction becomes explosive in the presence of linear damping of the waves. We note that the explosive instability studied by the well-defined phase approach is a first-order phenomenon in the order of nonlinear perturbation, and explosive instability may be developed to higher orders.

2. A brief review of the perturbation method

Coffey and Ford (1969) have presented a form of the method of averaging called the method of rapidly rotating phase. We consider the following set of coupled differential equations:

$$\mathrm{d}x_i/\mathrm{d}t = \epsilon A_i(X, \Psi), \qquad i = 1, 2, \ldots \gamma, \tag{1a}$$

$$\mathrm{d}\psi_j/\mathrm{d}t = \omega_j(X) + \epsilon Bj(X, \Psi), \qquad j = 1, 2, \ldots s, \tag{1b}$$

$$X = (x_1, x_2, \ldots x_\gamma),$$

$$\Psi = (\psi_1, \psi_2, \ldots \psi_s),$$

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where ϵ is a small parameter and X, Ψ and the A_i's and B_j's are periodic functions of the ψ_k's with period 2π. When $\epsilon = 0$, the x_i's are constant and the ψ_j's are linear functions of time.

When ϵ is small the x_i's will experience a slow secular growth with a small-amplitude rapid fluctuation superimposed on it. Similarly the ψ_j's will experience a rapid secular growth on which is superimposed a small-amplitude rapid fluctuation. The method is utilised to separate this secular motion from the rapidly fluctuating motion. To do this we seek a solution of the form

$$x_i = y_i + \sum_{n=1}^{\infty} \epsilon^n F_i^{(n)}(\phi), \qquad i = 1, 2, \ldots \gamma, \tag{2a}$$

$$\psi_j = \phi_j + \sum_{n=1}^{\infty} \epsilon^n G_j^{(n)}(\phi), \qquad j = 1, 2, \ldots s, \tag{2b}$$

where $F_i^{(n)}$ and $G_j^{(n)}$ are periodic functions of each of the ϕ_k, with period 2π.

We further require that y_i and ϕ_j should satisfy the following differential equations:

$$\begin{aligned} \mathrm{d}y_i/\mathrm{d}t &= \sum_{n=1}^{\infty} \epsilon^n a_i^{(n)}(y), \qquad i = 1, 2, \ldots \gamma, \\ \mathrm{d}\phi_j/\mathrm{d}t &= \omega_j(y) + \sum_{n=1}^{\infty} \epsilon^n b_j^{(n)}(y), \qquad j = 1, 2, \ldots s, \end{aligned} \tag{3}$$

where the right-hand sides of equations (3) are required to be independent of the ϕ_k's and the y_i's and ϕ_j's describe only secular motions since they are solutions of a system of differential equations which are independent of the rapidly changing phases ϕ_j. The rapid fluctuations of x_i and ψ_j about y_i and ϕ_j are given by the terms in the series in equation (2). We shall illustrate the working of the general method in the general system of three interacting waves in the presence of dissipation.

3. Basic coupled mode equations

A unified description of the nonlinear interaction of the waves can be made by the system of equations

$$\begin{aligned} \partial a_0/\partial t - \mathrm{i}\omega_0 a_0 &= c_{12}^* a_1 a_2, \\ \partial a_1/\partial t - \mathrm{i}\omega_1 a_1 &= c_{02} a_0 a_2^*, \\ \partial a_2/\partial t - \mathrm{i}\omega_2 a_2 &= c_{01} a_0 a_1^*. \end{aligned} \tag{4}$$

Following Wilhelmsson and Stenflo (1970) we can write

$$\begin{aligned} &a_j = \epsilon A_j(t) \exp(\mathrm{i}\,\mathrm{Re}\,\omega_j t), \qquad A_j = \tilde{u}_j \exp(\mathrm{i}\phi_j), \qquad e_{ij} = v_{ij} \exp(\mathrm{i}\theta_{ij}), \\ &\nu_j = \mathrm{Im}(\omega_j), \qquad \tilde{u}_j = |A_j|, \qquad v_{ij} = |c_{ij}|, \\ &\Delta\omega = \mathrm{Re}(\omega_0) - \mathrm{Re}(\omega_1) - \mathrm{Re}(\omega_2), \\ &\phi = \phi_0 - \phi_1 - \phi_2 + \Delta\omega t. \end{aligned} \tag{5}$$

We obtain the real system

$$
\begin{aligned}
\partial\tilde{u}_0/\partial t+\nu_0\tilde{u}_0 &= \epsilon v_{12}\tilde{u}_1\tilde{u}_2\cos(\phi+\theta_{12}),\\
\partial\tilde{u}_1/\partial t+\nu_1\tilde{u}_1 &= \epsilon v_{02}\tilde{u}_0\tilde{u}_2\cos(\phi+\theta_{02}),\\
\partial\tilde{u}_2/\partial t+\nu_2\tilde{u}_2 &= \epsilon v_{01}\tilde{u}_0\tilde{u}_1\cos(\phi+\theta_{01}),
\end{aligned}
\tag{6}
$$

$$
\frac{\partial\phi}{\partial t}=\Delta\omega-\epsilon v_{12}\frac{\tilde{u}_1\tilde{u}_2}{\tilde{u}_0}\sin(\phi+\theta_{12})-\epsilon v_{02}\frac{\tilde{u}_0\tilde{u}_2}{\tilde{u}_1}\sin(\phi+\theta_{02})-\epsilon v_{01}\frac{\tilde{u}_0\tilde{u}_1}{\tilde{u}_2}\sin(\phi+\theta_{01}).
$$

Using further renormalisations,

$$
\tilde{u}_0\to(v_{01}v_{02})^{1/2}u_0,\qquad \tilde{u}_1\to(v_{01}v_{12})^{1/2}u_1,\qquad \tilde{u}_2\to(v_{02}v_{12})^{1/2}u_2,
$$

we obtain

$$
\begin{aligned}
\partial u_0/\partial t+\nu_0 u_0 &= \epsilon u_1u_2\cos(\phi+\theta_{12}),\\
\partial u_1/\partial t+\nu_1 u_1 &= \epsilon u_0u_2\cos(\phi+\theta_{02}),\\
\partial u_2/\partial t+\nu_2 u_2 &= \epsilon u_0u_1\cos(\phi+\theta_{01}),
\end{aligned}
\tag{7a}
$$

$$
\frac{\partial\phi}{\partial t}=\Delta\omega-\epsilon\frac{u_1u_2}{u_0}\sin(\phi+\theta_{12})-\epsilon\frac{u_0u_2}{u_1}\sin(\phi+\theta_{02})-\epsilon\frac{u_0u_1}{u_2}\sin(\phi+\theta_{01}).\tag{7b}
$$

One may have explosively unstable solutions to equation (7*a*) when both amplitudes on the right-hand side of equation (7*a*) grow. This is possible only if all three amplitudes grow at the same time. In the next sections we shall discuss how the nonlinear three-wave interactions become explosive in the presence of linear damping and dissipation.

4. Effect of linear damping and dissipation on explosive instabilities of three interacting waves

To solve the set of equations (7*a*) and (7*b*) we find that the method of perturbation due to Coffey and Ford (1969) is the most suitable when $\Delta\omega\neq 0$, although it has limitations when $\Delta\omega=0$.

Following Coffey and Ford (1969) we seek a solution in the form

$$u_j=y_j+\epsilon F_j^{(1)}(\phi)+\epsilon^2F_j^{(2)}(\phi)+\ldots,\tag{8a}$$

$$\psi=\phi+\epsilon G^{(1)}(\phi)+\epsilon^2G^{(2)}(\phi)+\ldots,\tag{8b}$$

where

$$\dot{\phi}=\Delta\omega+\epsilon b^{(1)}(y)+\epsilon^2b^{(2)}(y)+\ldots,\tag{9a}$$

$$\dot{y}_j=a_j^{(0)}+\epsilon a_j^{(1)}(y)+\epsilon^2a_j^{(2)}(y)+\ldots.\tag{9b}$$

The $a_j^{(0)}$ term in equation (9*b*) appears because of the presence of the term with coefficient ν_j in equation (7*a*). Inserting equation (8) in equation (7), using equation (9) and equating powers of ϵ we obtain the following sequence of equations:

$$
\begin{aligned}
&a_j^{(0)} + \nu_j y_j = 0,\\
&a_0^{(1)} + (\partial F_0^{(1)}/\partial\phi)\Delta\omega + \nu_0 F_0^{(1)} = y_1 y_2 \cos(\phi+\theta_{12}),\\
&a_1^{(1)} + (\partial F_1^{(1)}/\partial\phi)\Delta\omega + \nu_1 F_1^{(1)} = y_0 y_2 \cos(\phi+\theta_{02}),\\
&a_2^{(1)} + (\partial F_2^{(1)}/\partial\phi)\Delta\omega + \nu_2 F_2^{(1)} = y_0 y_1 \cos(\phi+\theta_{01}),\\
&b^{(1)} + \frac{\partial G^{(1)}}{\partial\phi}\Delta\omega = -\left(\frac{y_1y_2}{y_0}\sin(\phi+\theta_{12}) + \frac{y_0y_2}{y_1}\sin(\phi+\theta_{02}) + \frac{y_0y_1}{y_2}\sin(\phi+\theta_{01})\right).
\end{aligned}
\tag{10}
$$

From the next power of ϵ we obtain

$$
\begin{aligned}
a_0^{(2)} &+ \frac{\partial F_0^{(2)}}{\partial\phi}\Delta\omega + \nu_0 F_0^{(2)} + b^{(1)}\frac{\partial F_0^{(1)}}{\partial\phi}\\
&= -y_1y_2G^{(1)}(\phi)\sin(\phi+\theta_{12}) + (y_1F_2^{(1)} + y_2F_1^{(1)})\cos(\phi+\theta_{12}),\\
a_1^{(2)} &+ \frac{\partial F_1^{(2)}}{\partial\phi}\Delta\omega + \nu_1 F_1^{(2)} + b^{(1)}\frac{\partial F_1^{(1)}}{\partial\phi}\\
&= -y_0y_2G^{(1)}(\phi)\sin(\phi+\theta_{02}) + (y_2F_0^{(1)} + y_0F_2^{(1)})\cos(\phi+\theta_{02}),\\
a_2^{(2)} &+ \frac{\partial F_2^{(2)}}{\partial\phi}\Delta\omega + \nu_2 F_2^{(2)} + b^{(1)}\frac{\partial F_2^{(1)}}{\partial\phi}\\
&= -y_0y_1G^{(1)}(\phi)\sin(\phi+\theta_{01}) + (y_1F_0^{(1)} + y_0F_1^{(1)})\cos(\phi+\theta_{01}),\\
b^{(2)} &+ \frac{\partial G^{(2)}}{\partial\phi}\Delta\omega\\
&= -[(y_2F_1^{(1)} + y_1F_2^{(1)})/y_0 - (y_1y_2/y_0^2)F_0^{(1)}]\sin(\phi+\theta_{12})\\
&\quad -[(y_2F_0^{(1)} + y_0F_2^{(1)})/y_1 - (y_0y_2/y_1^2)F_1^{(1)}]\sin(\phi+\theta_{02})\\
&\quad -[(y_1F_0^{(1)} + y_0F_1^{(1)})/y_2 - (y_0y_1/y_2^2)F_2^{(1)}]\sin(\phi+\theta_{01})\\
&\quad -G^{(1)}[(y_1y_2/y_0)\cos(\phi+\theta_{12}) + (y_0y_1/y_2)\cos(\phi+\theta_{01})\\
&\quad +(y_0y_2/y_1)\cos(\phi+\theta_{02})].
\end{aligned}
\tag{11}
$$

We solve the sequence of equations (7) and find

$$
a_j^{(1)} = 0, \qquad b^{(1)} = 0, \tag{12a}
$$

$$
\begin{aligned}
F_0^{(1)} &= (y_2y_1/\Delta\omega_0)\sin(\phi+\theta_{12}+\eta_0),\\
F_1^{(1)} &= (y_0y_2/\Delta\omega_1)\sin(\phi+\theta_{02}+\eta_1),\\
F_2^{(1)} &= (y_0y_1/\Delta\omega_2)\sin(\phi+\theta_{01}+\eta_2),
\end{aligned}
\tag{12b}
$$

$$
G^{(1)} = \frac{1}{\Delta\omega}\left(\frac{y_1y_2}{y_0}\cos(\phi+\theta_{12}) + \frac{y_0y_2}{y_1}\cos(\phi+\theta_{02}) + \frac{y_0y_1}{y_2}\cos(\phi+\theta_{01})\right),
\tag{12c}
$$

where

$$
\tan\eta_j = \nu_j/\Delta\omega_j, \qquad 1/\Delta\omega_j = (\nu_j^2 + \Delta\omega^2)^{-1/2}.
$$

One can obtain from equations (11)

$$a_0^{(2)}=\frac{y_0y_1^2}{2\Delta\omega_2}\sin(\theta_{01}-\theta_{12}+\eta_2)+\frac{y_0y_2^2}{2\Delta\omega_1}\sin(\theta_{02}-\theta_{12}+\eta_1)$$
$$-\frac{1}{2\Delta\omega}[y_0y_1^2\sin(\theta_{12}-\theta_{01})+y_0y_2^2\sin(\theta_{12}-\theta_{02})],$$
$$a_1^{(2)}=\frac{y_1y_0^2}{2\Delta\omega_2}\sin(\theta_{01}-\theta_{02}+\eta_2)+\frac{y_1y_2^2}{2\Delta\omega_0}\sin(\theta_{12}-\theta_{02}+\eta_0) \tag{13a}$$
$$-\frac{1}{2\Delta\omega}[y_1y_2^2\sin(\theta_{02}-\theta_{12})+y_1y_0^2\sin(\theta_{02}-\theta_{01})],$$
$$a_2^{(2)}=\frac{y_2y_0^2}{2\Delta\omega_1}\sin(\theta_{02}-\theta_{01}+\eta_1)+\frac{y_2y_1^2}{2\Delta\omega_0}\sin(\theta_{12}-\theta_{01}+\eta_0)$$
$$-\frac{1}{2\Delta\omega}[y_2y_1^2\sin(\theta_{01}-\theta_{12})+y_2y_0^2\sin(\theta_{01}-\theta_{02})],$$

$$\frac{\partial F_0^{(2)}}{\partial\phi}\Delta\omega+\nu_0F_0^{(2)}$$
$$=\frac{y_0y_1^2}{2\Delta\omega_2}\sin(2\phi+\theta_{01}+\theta_{12}+\eta_2)+\frac{y_0y_2^2}{2\Delta\omega_1}\sin(2\phi+\theta_{02}+\theta_{12}+\eta_1)$$
$$-\frac{1}{2\Delta\omega}\Bigg(y_0y_1^2\sin(2\phi+\theta_{12}+\theta_{01})+y_0y_2^2\sin(2\phi+\theta_{12}+\theta_{02})$$
$$+\frac{y_1^2y_2^2}{y_0}\sin 2(\phi+\theta_{12})\Bigg),$$

$$\frac{\partial F_1^{(2)}}{\partial\phi}\Delta\omega+\nu_1F_1^{(2)}$$
$$=\frac{y_1y_0^2}{2\Delta\omega_2}\sin(2\phi+\theta_{01}+\theta_{02}+\eta_2)+\frac{y_1y_2^2}{2\Delta\omega_0}\sin(2\phi+\theta_{12}+\theta_{02}+\eta_0)$$
$$-\frac{1}{2\Delta\omega}\Bigg(y_1y_0^2\sin(2\phi+\theta_{02}+\theta_{01})+y_1y_2^2\sin(2\phi+\theta_{02}+\theta_{12})$$
$$+\frac{y_0^2y_2^2}{y_1}\sin 2(\phi+\theta_{02})\Bigg),$$

$$\frac{\partial F_2^{(2)}}{\partial\phi}\Delta\omega+\nu_2F_2^{(2)}$$
$$=\frac{y_2y_0^2}{2\Delta\omega_1}\sin(2\phi+\theta_{02}+\theta_{01}+\eta_1)+\frac{y_2y_1^2}{2\Delta\omega_0}\sin(2\phi+\theta_{12}+\theta_{01}+\eta_0)$$
$$-\frac{1}{2\Delta\omega}\Bigg(y_2y_0^2\sin(2\phi+\theta_{01}+\theta_{02})+y_2y_1^2\sin(2\phi+\theta_{01}+\theta_{12})$$
$$+\frac{y_0^2y_1^2}{y_2}\sin 2(\phi+\theta_{01})\Bigg), \tag{13b}$$

$$b^{(2)}=\frac{y_1^2y_2^2}{2y_0^2}\left(\frac{\cos\eta_0}{\Delta\omega_0}-\frac{1}{\Delta\omega}\right)+\frac{y_0^2y_2^2}{2y_1^2}\left(\frac{\cos\eta_1}{\Delta\omega_1}-\frac{1}{\Delta\omega}\right)+\frac{y_0^2y_1^2}{2y_2^2}\left(\frac{\cos\eta_2}{\Delta\omega_2}-\frac{1}{\Delta\omega}\right)$$
$$-y_0^2\left(\frac{\cos(\theta_{02}-\theta_{01}+\eta_1)}{2\Delta\omega_1}+\frac{\cos(\theta_{01}-\theta_{02}+\eta_2)}{2\Delta\omega_2}+\frac{\cos(\theta_{02}-\theta_{01})}{\Delta\omega}\right)$$
$$-y_1^2\left(\frac{\cos(\theta_{01}-\theta_{12}+\eta_2)}{2\Delta\omega_2}+\frac{\cos(\theta_{12}-\theta_{01}+\eta_0)}{2\Delta\omega_0}+\frac{\cos(\theta_{12}-\theta_{01})}{\Delta\omega}\right)$$
$$-y_2^2\left(\frac{\cos(\theta_{02}-\theta_{12}+\eta_1)}{2\Delta\omega_1}+\frac{\cos(\theta_{12}-\theta_{02}+\eta_0)}{2\Delta\omega_0}+\frac{\cos(\theta_{12}-\theta_{02})}{\Delta\omega}\right),$$

$$G^{(2)}=\frac{1}{4\Delta\omega}\Bigg[\Bigg(\frac{y_1^2}{\Delta\omega_2}\sin(2\phi+\theta_{01}+\theta_{12}+\eta_2)+\frac{y_2^2}{\Delta\omega_1}\sin(2\phi+\theta_{12}+\theta_{02}+\eta_1)$$
$$+\frac{y_0^2}{\Delta\omega_2}\sin(2\phi+\theta_{01}+\theta_{02}+\eta_2)+\frac{y_2^2}{\Delta\omega_0}\sin(2\phi+\theta_{12}+\theta_{02}+\eta_0)$$
$$+\frac{y_0^2}{\Delta\omega_1}\sin(2\phi+\theta_{01}+\theta_{02}+\eta_1)+\frac{y_1^2}{\Delta\omega_0}\sin(2\phi+\theta_{01}+\theta_{12}+\eta_0)\Bigg)$$
$$-\left(\frac{y_1^2y_2^2}{y_0^2\Delta\omega_0}\sin(2\phi+2\theta_{12}+\eta_0)+\frac{y_0^2y_2^2}{y_1^2\Delta\omega_1}\sin(2\phi+2\theta_{02}+\eta_1)\right.$$
$$\left.+\frac{y_0y_1^2}{y_2^2\Delta\omega_2}\sin(2\phi+2\theta_{01}+\eta_2)\right)$$
$$-\frac{1}{\Delta\omega}\left(\frac{y_1^2y_2^2}{y_0^2}\sin 2(\phi+\theta_{12})+\frac{y_0^2y_2^2}{y_1^2}\sin 2(\phi+\theta_{02})+\frac{y_0^2y_1^2}{y_2^2}\sin 2(\phi+\theta_{01})\right)$$
$$-(2/\Delta\omega)[y_0^2\sin(2\phi+\theta_{02}+\theta_{01})+y_1^2\sin(2\phi+\theta_{12}+\theta_{01})$$
$$+y_2^2\sin(2\phi+\theta_{12}+\theta_{02})]\Bigg]. \tag{13c}$$

4.1. *The dissipation-free case*

For the dissipation-free case the angles θ_{ij} will either be the same (explosively unstable case) or differ by π (stable case). When all θ_{ij} are the same and we assume further that $\nu_0=\nu_1=\nu$ and $\nu_2=0$, then from equations (9*b*) and equations (13*a*) we obtain

$$\dot{y}_0+\nu y_0=(y_0y_2^2\sin\eta)/2\Delta\omega_k \qquad (\eta_1=\eta_0=\eta,\ \eta_2=0,\ \Delta\omega_0=\Delta\omega_1=\Delta\omega_k),$$
$$\dot{y}_1+\nu y_1=(y_1y_2^2\sin\eta)/2\Delta\omega_k, \tag{14a}$$
$$\dot{y}_2=(y_1y_0^2\sin\eta)/2\Delta\omega_k+(y_2y_1^2\sin\eta)/2\Delta\omega_k.$$

We can write

$$y_j=Y_j\,e^{-\nu t},\qquad x_j=Y_j^2,\qquad \lambda=\sin\eta/\Delta\omega_k$$

to obtain

$$d(x_2-x_0-x_1)/dt=0, \tag{14b}$$
$$d(\log x_2)/dt=\lambda(x_0+x_1). \tag{14c}$$

Equations (14*b*) and (14*c*) can be integrated to obtain

$$x_2 = P(\mu_+ + \mu_- \, e^{\lambda P t})/(\mu_+ - \mu_- \, e^{\lambda P t}),$$
$$x_0 = x_1 = A \, e^{(P\lambda - 1)t}/(\mu_+ - \mu_- \, e^{\lambda P t})^{1/\lambda P},$$

where $\mu_\pm = x_2(0) \pm P$, $P = x_2(0) - x_1(0) - x_0(0)$ and A is a constant. All x_j will go to infinity and the time of explosion will be

$$t_\infty = \frac{1}{\lambda P} \log \frac{\mu_+}{\mu_-}.$$

If we choose the initial amplitudes such that $P = 0$ we obtain the simplest possible form

$$x_2(t) = \frac{1}{1/x_2(0) - \lambda t},$$

and thus the time of explosion $t_\infty = 1/\lambda x_2(0)$. In the presence of damping $\lambda < 1$, which shows that the time of explosion will be delayed. In an exactly similar manner one can obtain that the x_j have a stable solution when the θ_{ij} differ by π.

4.2. Influence of dissipation when θ_{ij} are present

Using (12*a*) the differential equation (9*a*) becomes, to second order in ϵ with $x_j = y_j^2$,

$$\begin{aligned}
\frac{dx_0}{dt} + 2\nu_0 x_0 &= \epsilon^2 \Big[x_0 x_1 \Big(\frac{1}{\Delta\omega_2} \sin(\theta_{01} - \theta_{12} + \eta_2) + \frac{1}{\Delta\omega} \sin(\theta_{01} - \theta_{12}) \Big) \\
&\quad + x_0 x_2 \Big(\frac{1}{\Delta\omega_1} \sin(\theta_{02} - \theta_{12} + \eta_1) + \frac{1}{\Delta\omega} \sin(\theta_{02} - \theta_{12}) \Big) \Big], \\
\frac{dx_1}{dt} + 2\nu_1 x_1 &= \epsilon^2 \Big[x_0 x_1 \Big(\frac{1}{\Delta\omega_2} \sin(\theta_{01} - \theta_{02} + \eta_2) + \frac{1}{\Delta\omega} \sin(\theta_{01} - \theta_{02}) \Big) \\
&\quad + x_1 x_2 \Big(\frac{1}{\Delta\omega_0} \sin(\theta_{12} - \theta_{02} + \eta_0) + \frac{1}{\Delta\omega} \sin(\theta_{12} - \theta_{02}) \Big) \Big], \\
\frac{dx_2}{dt} + 2\nu_2 x_2 &= \epsilon^2 \Big[x_0 x_2 \Big(\frac{1}{\Delta\omega_1} \sin(\theta_{02} - \theta_{01} + \eta_1) + \frac{1}{\Delta\omega} \sin(\theta_{02} - \theta_{01}) \Big) \\
&\quad + x_1 x_2 \Big(\frac{1}{\Delta\omega_0} \sin(\theta_{12} - \theta_{01} + \eta_0) + \frac{1}{\Delta\omega} \sin(\theta_{12} - \theta_{01}) \Big) \Big].
\end{aligned} \tag{15a}$$

Assuming all ν_j are the same, we have $\eta_0 = \eta_1 = \eta_2 = \eta$, also assuming $\Delta\omega \gg \nu$, and η much less than the difference of the θ_{ij}. One can derive the following constants of motion from equation (15*a*):

$$\frac{d}{d\tau} (\log x_0 + \log x_1 + \log x_2) = 0, \tag{15b}$$

$$\frac{d}{d\tau} [x_0 \sin(\theta_{01} - \theta_{02}) + x_1 \sin(\theta_{12} - \theta_{01}) + x_2 \sin(\theta_{02} - \theta_{12})] = 0, \tag{15c}$$

where

$$x_j = x_j \, e^{2\nu t}, \qquad \tau = (1 - e^{-2\nu t})/2\nu.$$

Equations (15*a*) can be called the generalised Volterra equations (Hirota 1976). It is interesting to note that the constants of motion (15*b*) are very similar to the condition of equilibrium obtained from the entropy function (Dikasov *et al* 1965) and that equation (15*c*) is identical to that of Wilhelmsson and Stenflo (1970).

The method of nonlinear perturbation (Coffey and Ford 1969) does not in general lead to an explicit solution of the original set of equations. It is a method very suitable for separation of the secular motion from the rapid periodic fluctuation and for reducing the problem to that of solving the differential equations for secular motion alone. We proceed to solve the differential equations for secular motion (equations (9*a*) and (9*b*)) for different orders of perturbation. We write

$$P = x_0(0)\sin(\theta_{01}-\theta_{02}) + x_1(0)\sin(\theta_{12}-\theta_{01}) + x_2(0)\sin(\theta_{02}-\theta_{12}),$$
$$Q = x_0(0)x_1(0)x_2(0).$$

Equations (15*a*) can be written as

$$\int_{x_0(0)}^{x_0(\tau)} \frac{\mathrm{d}x_0}{2\{(x_0^2/\Delta\omega^2)[P - x_0\sin(\theta_{01}-\theta_{02})]^2 - (4Qx_0/\Delta\omega^2)\sin(\theta_{12}-\theta_{01})\sin(\theta_{02}-\theta_{12})\}^{1/2}} = \int_0^\tau \epsilon^2\,\mathrm{d}\tau,$$

$$\int_{x_1(0)}^{x_1(\tau)} \frac{\mathrm{d}x_1}{2\{(x_1^2/\Delta\omega^2)[P - x_1\sin(\theta_{12}-\theta_{01})]^2 - (4Qx_1/\Delta\omega^2)\sin(\theta_{02}-\theta_{12})\sin(\theta_{01}-\theta_{02})\}^{1/2}} = \int_0^\tau \epsilon^2\,\mathrm{d}\tau,$$

$$\int_{x_2(0)}^{x_2(\tau)} \frac{\mathrm{d}x_2}{2\{(x_2^2/\Delta\omega^2)[P - x_2\sin(\theta_{02}-\theta_{12})]^2 - (4Qx_2/\Delta\omega^2)\sin(\theta_{02}-\theta_{12})\sin(\theta_{12}-\theta_{01})\}^{1/2}} = \int_0^\tau \epsilon^2\,\mathrm{d}\tau. \quad (16)$$

Equations (16) are in general elliptic integrals and the solution can be expressed in terms of elliptic functions, depending on whether the solution of $\pi(x_j)=0$ has either four real roots or two real and two complex roots (Weiland and Wilhelmsson 1977). We notice that the solution of $\pi(x_j)=0$ ($\pi(x_j)$ stands for the denominator of equation (16)) has (a) two real roots and two complex roots when $\theta_{02} > \theta_{12} > \theta_{01}$; (b) four real roots when any two of three θ_{ij} are equal.

The significant change one can note is that $\pi(x_j)$ is in general biquadratic, while in the dissipation-free case in the absence of linear damping it is cubic. The general solution has been discussed in the Appendix.

For simplicity let us take the initial values of x_j and θ_{ij} such that when $P=0$ all the $\pi(x_j)$ of equations (16) have two real and two complex roots. The integrals of equations (16) can be written in the form (see Appendix)

$$\int_{x_j(0)}^{x_j(\tau)} \frac{\mathrm{d}x_j}{(\pi(x_j))^{1/2}} = \frac{g_j - f_j}{\sqrt{A_j}} \int_{u_j(0)}^{u_j(\tau)} \frac{\mathrm{d}u_j}{[(1+m_j u_j^2)(1+n_j u_j^2)]^{1/2}}, \quad (17a)$$

with

$$x_j = \frac{f_j + g_j u_j}{1+u_j}, \qquad f_j = -\frac{(\sqrt{3}+1)}{2}\alpha_j, \qquad g_j = \frac{(\sqrt{3}-1)}{2}\alpha_j,$$
$$\alpha_j = (\sin\psi_j)^{-1}(4Q\sin\psi_0\sin\psi_1\sin\psi_2)^{1/3}, \qquad j=0,1,2,$$
$$\psi_0 = \theta_{01}-\theta_{02}, \qquad \psi_1 = \theta_{12}-\theta_{01}, \qquad \psi_2 = \theta_{02}-\theta_{12},$$
$$n_j = 1, \qquad A_j = (3\sqrt{3}/4)\alpha_j^4(\sqrt{3}+2),$$
$$m_j = -(\sqrt{3}+2)^{-2} = -\cot^2\theta.$$

Then equations (16) reduce to

$$\int_{u_j(0)}^{u_j(\tau)} \frac{\mathrm{d}u_j}{[(1+m_j u_j)^2(1+n_j u_j^2)]^{1/2}} = \frac{\sqrt{A_j}}{g_j - f_j}\frac{2\epsilon^2}{\Delta\omega}\int_0^\tau \mathrm{d}\tau. \tag{18}$$

Integrating equation (18) (Abramowitz and Stegun 1965) one obtains

$$u_j = \sin\theta\ \mathrm{sd}\left\{\left[\frac{\sqrt{A_j}}{g_j-f_j}\left(\frac{2\epsilon^2}{\Delta\omega}\sin\psi_j\right)\operatorname{cosec}\theta(\tau+\tau_j)\right]\Big|\sin^2\theta\right\} \tag{19}$$

(where sd is Jacobi's elliptic function) where τ_j is defined as

$$\tau_j = \frac{g_j - f_j}{\sqrt{A_j}}\frac{\sin\theta}{[(2\epsilon^2/\Delta\omega)\sin\psi_j]}\mathrm{sd}^{-1}\left(\frac{u_j(0)}{\sin\theta}\Big|\sin^2\theta\right).$$

When $u_j(\tau) = -1$, x_j tends to infinity and one can obtain from equation (19) the time of explosion given by

$$\sin\theta\ \mathrm{sd}\left[\frac{\sqrt{A_j}}{g_j-f_j}\left(\frac{2\epsilon^2}{\Delta\omega}\sin\psi_j\right)\operatorname{cosec}\theta(\tau+\tau_j)\Big|\sin^2\theta\right]+1=0. \tag{20}$$

When, in addition to $P=0$, the initial conditions of $x_j(0)$ and θ_{ij} are such that the $u_j(0)$ are given by $u_j(0) = \sin(\theta+\pi)\,\mathrm{sd}(l|\sin^2\theta)$ with l a constant, all the x_j will grow to infinity at the same time and the time of explosion will be obtained as

$$\tau_\infty = 3^{-1/4}\frac{\Delta\omega}{2\epsilon^2}[l - \mathrm{sd}^{-1}(\operatorname{cosec}\theta|\sin^2\theta)].$$

5. Discussion

In this section we discuss some current literature on explosive instabilities (see Fukai *et al* (1970, 1971), Aamodt and Sloan (1967, 1968)) relevant to our present work. They have investigated this phenomenon by deriving the equations for the time evolution of the complex wave amplitude, retaining only the second-order nonlinear interaction term. In an attempt to obtain more physically acceptable results, several authors (Fukai *et al* (1970), Dysthe (1970), Oraevskii *et al* (1973a, b), Weiland and Wilhelmsson (1977)) retained the third-order nonlinear terms in the equations for the complex amplitudes to obtain new coupled mode equations which in particular cases were amenable to analysis.

The insufficiency of the first-order approximation for a resonant wave interaction and in a multistream plasma has been well discussed by Sedlacek (1975a, b, 1976), who used the theory of nearly multiple periodic Hamiltonian systems (Coffey 1969, Sedlacek 1975a, b).

In our paper we have used the nonlinear perturbation technique developed by Coffey and Ford (1969) to separate the fast oscillations from the slow evolution of the whole system. This is a significant advantage over the time-averaging scheme of Bogolyubov and Krylov and Mitropolski, and over the method of averaged Lagrangians as elaborated by Dougherty (1970) and applied by Galloway and Kim (1971) and Boyd and Turner (1972, 1973).

Another main advantage of our method lies in the fact that analysis of the explosive and stabilised nature of three-wave interactions is possible from the explicit solutions of the secular motion.

It is interesting to note that the nonlinear coupled mode equation (equation (7)) is reproduced from the first-order equation (equation (10)) of our analysis when all F and G are taken to be zero. Because of this coincidence, one may conclude that our first-order approximation is exactly equivalent to the usual time-averaging scheme.

A new set of constants of motion is obtained in the second-order approximation. If one wishes, one can study the character of motion by the phase plane analysis (Minorsky 1962). In such a description, when a curve is drawn with the amplitude as independent variable and the amplitude derivative as dependent variable (see equation (15a)), one obtains the phase portraits of explosive instability. Again the effect of inclusion of the phases θ_{ij} and $\Delta\omega$ may introduce a large negative root of $\pi(x_j)=0$ (equation (16)). In such cases, in the second-order approximation, the motion will be stabilised. The significant influence of the third-order nonlinear term, having a similar effect, has been discussed by Weiland and Wilhelmsson (1977) and Byers *et al* (1971).

In our analysis, it has been shown how in the second-order approximation the explosive character of three-wave nonlinear interactions is greatly influenced by the presence of linear damping and dissipation. The same order of nonlinearity that causes explosive instability in the first-order approximation may stabilise the waves in the second-order approximation.

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Appendix

An integral of the form $\int dz/(\pi(z))^{1/2}$ can be expressed in terms of an elliptic integral when $\pi(z)$ can be written as

$$\pi(z)=(z^2+pz+q)(z^2+rz+s) \tag{A1}$$

where p, q, r, s are real. In the transformation $z=(f+gu)/(1+u)$ let f, g, be so chosen that the coefficient of u in each quadratic is zero; then $\pi(z)$ will take the form

$$\pi(z)=A(1+mu^2)(1+nu^2)/(1+u)^4, \tag{A2}$$

with

$$A=(f^2+pf+q)(f^2+rf+s),$$

$$m=\frac{g^2+pg+q}{f^2+pf+q}, \qquad n=\frac{g^2+rg+s}{f^2+rf+s}, \tag{A3}$$

$$f+g=\frac{2(q-s)}{r-p}, \qquad fg=\frac{ps-qr}{r-p}.$$

Now let the two equations

$$z^2+pz+q=0 \quad \text{and} \quad z^2+rz+s=0 \tag{A4}$$

have roots x_1, x_2 and x_3, x_4 respectively so that

$$x_1+x_2=-p, \qquad x_1x_2=q,$$

$$x_3+x_4=-r, \qquad x_3x_4=s.$$

Further, f and g are the roots of the equation

$$(r-p)f^2+2(s-q)f+(ps-qr)=0.$$

Accordingly the roots will be real when

$$(s-q)^2-(r-p)(ps-qr)>0. \tag{A5}$$

The inequality can be written as

$$(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)>0. \tag{A6}$$

This inequality holds when at least one of equations (A4) has imaginary roots. If both equations have two real roots the factors of $\pi(z)$ can always be written so that $x_1>x_2>x_3>x_4$.

Thus the inequality holds in this case also. Then the integral can be reduced to an elliptic integral (Abramowitz and Stegun 1965):

$$\int\frac{\mathrm{d}z}{(\pi(z))^{1/2}}=\frac{g-f}{\sqrt{A}}\int_{u_0}^{u}\frac{\mathrm{d}u}{[(1+mu^2)(1+nu^2)]^{1/2}}. \tag{A7}$$

For the case $m<0$, $n>0$

$$\int\frac{\mathrm{d}z}{(\pi(z))^{1/2}}=\frac{g-f}{(Amn)^{1/2}}\left[\int_0^u\frac{\mathrm{d}u}{[(1/m-u^2)(1/n+u^2)]^{1/2}}-\int_0^{u_0}\frac{\mathrm{d}u}{[(1/m-u^2)(1/n+u^2)]^{1/2}}\right]$$

$$=\frac{g-f}{(A(m+n))^{1/2}}\left[\mathrm{sd}^{-1}\left(\frac{u(m+n)^{1/2}}{mn}\,\Big|\,\frac{m}{m+n}\right)-\mathrm{sd}^{-1}\left(\frac{u_0(m+n)^{1/2}}{mn}\,\Big|\,\frac{m}{m+n}\right)\right]$$

where sd^{-1} is Jacobi's inverse elliptic function.

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